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Mark J. Schervish; Teddy Seidenfeld; Joseph B. Kadane

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SHARED PREFERENCES AND STATE-DEPENDENT UTILITIES*

MARK J. SCHERVISH, TEDDY SEIDENFELD AND JOSEPH B. KADANE
Department of Statistics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213
Department of Philosophy, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213
Department of Statistics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213

This investigation combines two questions for expected utility theory:

1. When do the shared preferences among expected utility maximizers conform to the dictates of expected utility?
2. What is the impact on expected utility theory of allowing preferences for prizes to be state-dependent?

Our principal conclusion (Theorem 4) establishes very restrictive necessary and sufficient conditions for the existence of a Pareto, Bayesian compromise of preferences between two Bayesian agents, even when utilities are permitted to be state-dependent and identifiable. This finding extends our earlier result (Theorem 2, 1989a) which applies provided that all utilities are state-independent. A subsidiary theme is a decision theoretic analysis of common rules for "pooling" expert probabilities.

Against the backdrop of "horse lottery" theory (Anscombe and Aumann 1963) and subject to a weak Pareto rule, we show, generally, that there is no Bayesian compromise between two Bayesian agents even when state-dependent utilities are entertained in an *identifiable* way. The word "identifiable" is important because, if state-dependence is permitted merely by dropping the Anscombe-Aumann axiom (Axiom 4 here) for "state-independence," though a continuum of possible Bayesian compromises emerges, also it leads to an extreme underdetermination of an agent's personal probability and utility given the agent's preferences. Instead, when state-dependence is monitored through (our version of) the approach of Karni, Schmeidler, and Vind (1983), the general impossibility of a Bayesian, Pareto compromise in preferences reappears.

(CONSENSUS; HORSE LOTTERIES; PRIZE-STATE LOTTERIES; SUBJECTIVE EXPECTED UTILITY)

1. Introduction

This paper combines two questions for expected utility theory:

1. When do the shared preferences among expected utility maximizers conform to the dictates of expected utility?
2. What is the impact on expected utility theory of allowing preferences for prizes to be state-dependent?

Against the backdrop of the "horse lottery" theory of Anscombe and Aumann (1963) and subject to a weak Pareto rule, we show that, in general, there is no Bayesian compromise between two Bayesian agents even when state-dependent utilities are entertained in an *identifiable* way. In order to see the relevance in pairing these two issues, first consider the importance of each question alone.

Regarding shared preferences, it is natural to inquire whether justifications of expected utility theory can be extended from a single agent to a cooperative group in such a way as to preserve those preferences common to them all. For an illustration, consider two coherent decision makers, call them Dick and Jane, who wish to act in unison—with binding agreements possible—in a fashion that their collective choices conform to axiomatic canons of expected utility theory. This is, suppose

- (i) whenever both Dick and Jane (separately) think that option A_2 is strictly better

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than option A_1 , then A_1 cannot be their cooperative choice if A_2 is available—a (weak) Pareto condition on cooperative, group preference; and suppose

(ii) their collective choices are coherent.

Then the first question amounts to asking when there is a Pareto compromise of individual preferences that satisfies the axiomatic constraints which constitute “coherence.”

Seidenfeld, Kadane, and Schervish (1989a) examines this question for the Horse Lottery theory of rationality proposed by Anscombe and Aumann (1963). Their theory, like many others (such as Savage 1954), axiomatizes rational preferences for acts defined as functions from states to outcomes. Anscombe and Aumann’s theory is distinguished by its use of simple von Neumann-Morgenstern lotteries for outcomes. (A *von Neumann-Morgenstern lottery*, denoted by L , is a probability distribution over a set \mathcal{G} of prizes. A lottery is *simple* if its distribution has finite support. All lotteries discussed in this article are simple.) Anscombe-Aumann acts, called *horse lotteries* (denoted by H), are functions from states to von Neumann-Morgenstern lotteries. That is, let the set of states be \mathcal{X} , a finite set. For each $x \in \mathcal{X}$ and each horse lottery H , $H(x)$ is the von Neumann-Morgenstern lottery which H provides when state x occurs. Thus, in contrast with Savage’s theory, Horse Lottery theory relies on an extraneous account of probability for defining outcomes (and applies just for simple acts: acts which assume only finitely many lottery outcomes).

The use of extraneous probability affords an elegant axiomatization of rational preference for horse lotteries. Let \prec (\cong) denote, respectively, strict (weak) preference between acts. For $0 \leq \alpha \leq 1$, denote by $\alpha H_1 + (1 - \alpha)H_2$, defined (as in the theory of von Neumann and Morgenstern 1947) as the state-by-state convex combination of their corresponding lottery outcomes. Last, let H_{L_i} denote the “constant” horse lottery that awards the lottery L_i in every state. Four axioms summarize Anscombe-Aumann theory for nontrivial preferences. (Note that the first three axioms comprise the von Neumann-Morgenstern Utility theory.)

AXIOM 1. Preference is a weak order. *That is, \leq is reflexive and transitive, with every pair of acts compared.*

AXIOM 2. Independence. *For every H_1, H_2, H_3 , and every $0 < \alpha \leq 1$,*

$$H_1 \leq H_2 \quad \text{if and only if} \quad \alpha H_1 + (1 - \alpha)H_3 \leq \alpha H_2 + (1 - \alpha)H_3.$$

AXIOM 3. Archimedes. *If $H_1 \prec H_2 \prec H_3$, there exist $0 < \alpha, \beta < 1$, such that*

$$\alpha H_1 + (1 - \alpha)H_3 \prec H_2 \prec \beta H_1 + (1 - \beta)H_3.$$

The next axiom refers to *nonnull* states.

DEFINITION 1. A state x^* is *null* for an agent if he/she is indifferent between each pair of acts that have the same outcomes (state-by-state) on the remaining states $x \neq x^*$. That is, if x^* is null, it does not matter to the agent’s assessment of acts what outcome results when x^* occurs. A state is *nonnull* if it is not null.

AXIOM 4. State-independent preference for lotteries. *Given two von Neumann-Morgenstern lotteries L_1 and L_2 , let H_1 and H_2 be two horse lotteries which differ only in that, for some nonnull state x^* , $H_1(x^*) = L_1$ and $H_2(x^*) = L_2$ ($H_1(x) = H_2(x)$ for $x \neq x^*$). (That is, the two acts, H_1 and H_2 are (state by state) identical except for state x^* .) Then*

$$H_{L_1} \leq H_{L_2} \quad \text{if and only if} \quad H_1 \leq H_2.$$

In words, Axiom 4 requires that the agent’s preference for outcomes (where outcomes can be thought of also as constant acts) replicate under each state for acts that are “called off,” except in that state. Throughout this paper, we will assume that preferences are not trivial. That is, each agent holds some strict preferences. This corresponds to Savage’s

postulate P5. The four Horse lottery axioms have natural counterparts within Savage's theory for simple acts, P1–P6: Axioms 1 and 2 correspond, respectively, to Savage's postulates P1 and P2. The Archimedean Axiom 3 is contained within Savage's technical P6. Last, Savage's P3 (and part of P4) serve the same purposes as Axiom 4.

The next theorem introduces a utility function to be thought of as a function U from the set of prizes \mathcal{G} to the real numbers. If L is a von Neumann-Morgenstern lottery corresponding to the simple probability distribution Q over \mathcal{G} , then we write $U(L)$ to stand for $\sum_{\mathcal{G}} U(g)Q(g)$.

THEOREM 1 (Anscombe and Aumann 1963). *Axioms 1–4 are satisfied if and only if there exist a (state-independent) utility U (unique up to positive affine transformation) over prizes and a unique personal probability P over states (with $P(s) > 0$ if and only if s is nonnull), satisfying, for every H_1 and H_2 ,*

$$H_1 \leq H_2 \quad \text{if and only if} \quad \sum_j P(x_j)U(L_{1j}) \leq \sum_j P(x_j)U(L_{2j}).$$

That is, rational preference according to Anscombe-Aumann theory is equivalent to expected utility theory with a personal probability P over states and a utility U over prizes.

The central result of Seidenfeld, Kadane, and Schervish (1989a) about shared preferences for two agents, Dick and Jane, whose preferences satisfy these four axioms, is as follows. Let \leq_D (\leq_J) be, respectively, Dick's (Jane's) preference. By the previous theorem, each preference order is summarized by the probability/utility pair (P_D, U_D) or (P_J, U_J) . Suppose these two decision makers have different personal degrees of belief, $P_D \neq P_J$, and different utility functions for prizes, $U_D \neq U_J$. And suppose there are two prizes which they rank order the same. (See footnote 11 in Seidenfeld, Kadane, and Schervish 1989a for a discussion of the significance of this assumption.) Let \prec be the strict partial order created with the (weak) Pareto condition, discussed above. That is, define $H_1 \prec H_2$ if and only if, both $H_1 \prec_D H_2$ and $H_1 \prec_J H_2$, i.e., if and only if Dick and Jane each prefers H_2 over H_1 .

THEOREM 2 (Seidenfeld, Kadane, and Schervish 1989a). *The partial order \prec agrees with no coherent preference \leq except for the two agents' preferences. That is, except for \leq_D and \leq_J , no preferences satisfy the Anscombe-Aumann theory while preserving the strict preferences captured by \prec .*

In other words, there is no coherent (Pareto) compromise of preferences available to Dick and Jane. See Seidenfeld, Kadane, and Schervish (1989a) for discussions of the relation between Theorem 2 and the celebrated Possibility Theorem of Arrow (1951), and for the relation to important papers by Hylland and Zeckhauser (1979) and Hammond (1981). The negative conclusion of Theorem 2 is reminiscent, also, of recent syndicate-theoretic work by Pratt and Zeckhauser (1989). That theory puts very restrictive conditions on the extent to which agents in a coherent syndicate may hold different beliefs or utilities. The principal contrast with Theorem 2, however, is that the syndicate is formulated with individual veto rights. Each member of the syndicate may exercise a veto whenever an option is judged by that agent to be inferior to the "status quo." Theorem 2 operates under the weak Pareto rule, without an additional veto authority for individuals.

The import of the negative result in Theorem 2 depends, of course, upon the adequacy of Horse Lottery theory as an account of expected utility theory. For instance, it is easy to show (see footnote 12 of Seidenfeld, Kadane, and Schervish 1989a) that if Axiom 4 is dropped, there are a continuum of Pareto compromises satisfying the first three Horse Lottery axioms. Thus, we arrive at the second of the two questions posed in the opening paragraph: What is the significance of Axiom 4 for expected utility theory?

We begin our answer by reviewing a formally trivial point. Let the preference order \leq be represented by expected utility, using a (possibly) *state-dependent* utility. That is, suppose

$$H_1 \leq H_2 \quad \text{if and only if} \quad \sum_j P(x_j)U_j(H_1(x_j)) \leq \sum_j P(x_j)U_j(H_2(x_j)),$$

where it may be that, e.g., $U_2(g) \neq U_3(g)$ for some prizes g . It may be, contrary to Axiom 4, that the agent’s valuation for a particular prize depends upon the state in which it occurs.

Next, let P^* be a probability which is mutually absolutely continuous with P , i.e., P and P^* agree on the “null” states of 0 probability. Define the state-dependent utility U_j^* , by

$$U_j^*(g) = U_j(g)P(x_j)/P^*(x_j). \tag{1}$$

It follows immediately that

$$H_1 \leq H_2 \quad \text{if and only if} \quad \sum_j P^*(x_j)U_j^*(H_1(x_j)) \leq \sum_j P^*(x_j)U_j^*(H_2(x_j)).$$

Thus, using state-dependent utilities, personal probability is *wholly undefined* (up to null states) when probability and utility are reduced to a preference \leq over acts. Given only the preference structure \leq , its expected utility representation by a probability/utility pair is maximally underdetermined. (See Schervish, Seidenfeld, and Kadane 1990 for a discussion of state-dependence as it applies to the theories of Anscombe and Aumann 1963 and Savage 1954.)

What is called for, then, is an expected utility theory that does not require Axiom 4 but instead introduces state-dependence in an identifiable way. To duplicate the intent of Theorem 1, we need extra “data” about preferences in order to carry out the measurement of probability and utility, where the latter may be state-dependent. Both of these goals are met in proposals by Karni, Schmeidler, and Vind (1983), elaborated by Karni (1985). In §7 of Schervish, Seidenfeld, and Kadane (1990), we simplify their construction.

The underlying theme is straightforward. The extra data on preferences, needed for investigating state-dependent utilities, involve a comparison of outcomes across states. One way to obtain these data (as suggested by Karni, Schmeidler, and Vind 1983) calls for a von Neumann-Morgenstern construction over prize-state outcomes. That is, the agent is asked to rank order (new kinds of hypothetical) lotteries over outcomes s , but where the state (x_j) is a component of that outcome. A *prize-state lottery* is just a stipulated probability distribution over the set of prize-state pairs (just as von Neumann-Morgenstern lotteries are stipulated probability distributions over the set of prizes). We will denote prize-state lotteries \hat{L} where $\hat{L}(g, x)$ is the stipulated probability of prize-state pair (g, x) .

The upshot of this approach is that the agent provides *two* preference orders: \leq over the original horse lottery acts and \sqsubseteq over prize-state lotteries. That is, unfortunately, the lotteries used to indicate state-dependent preferences are not merged with the horse lotteries. They are not part of a single preference order. The second order (\sqsubseteq) reveals the agent’s utility U for outcomes, which may be state-dependent. That utility is then used to restrict the potential expected-utility representations for the first order (\leq). The analysis is facilitated by a requirement that, for each nonnull state x , the agent’s preferences for prize-state lotteries involving only state x must agree with his/her “called-off” preferences for horse lotteries that agree on all but state x . This “consistency” condition between the two preference schemes, \leq and \sqsubseteq , is captured in Axiom 5, from Schervish, Seidenfeld,

and Kadane (1990). The version stated here allows for situations with arbitrarily many states.

AXIOM 5. Consistency. *Suppose the following conditions hold:*

- x^* is a nonnull state.
- For each pair \hat{L}_1 and \hat{L}_2 of prize-state lotteries whose probability distributions Q_1 and Q_2 assign probability 1 to the set of pairs $\mathcal{G} \times \{x^*\} = \{(g, x^*) : g \in \mathcal{G}\}$ (that is, with x^* fixed).
- L_1 and L_2 are the von Neumann-Morgenstern lotteries whose simple probability distributions over \mathcal{G} respectively agree with Q_1 and Q_2 when \mathcal{G} is thought of as equivalent to $\mathcal{G} \times \{x^*\}$.
- H_1 and H_2 are horse lotteries which satisfy $H_1(x^*) = L_1$, $H_2(x^*) = L_2$ and $H_1(y) = H_2(y)$ for all states $y \neq x^*$.

Then $\hat{L}_1 \sqsubseteq \hat{L}_2$ if and only if $H_1 \leq H_2$.

The question we answer in §2 is this: When state-dependent preferences are entertained (as we argue they ought to be), under what conditions will there be an expected-utility model for the shared preferences of two agents? That is, when can two agents acting cooperatively find a coherent, Pareto compromise of their two preference schemes, \leq_i and \sqsubseteq_i , $i = 1, 2$? We show that the “impossibility” reported in Theorem 2 obtains (with some minor qualifications).

We note, in passing, that our proof of Theorem 2 (from Seidenfeld, Kadane, and Schervish 1989a) does not facilitate a reduction of the state-dependent case to the state-independent case. We see the idea for the reduction prompted by Theorem 13.2, p. 177 of Fishburn (1970). Provided that there are (at least) two constant acts (of unequal value and with state-independent values) to serve as the 0 and 1 utility benchmarks across states, Fishburn’s result shows that the conclusion of Theorem 1 may be obtained even when other prizes are available only in designated states. Then, except for the two constant acts (0 and 1), we may as well say that each state has its own set of prizes, disjoint from every other state. The upshot is a version of Theorem 1 in which (except for the two constant acts) utility is vacuously state-independent. The reduction from state-dependence to state-independence occurs by declaring that (apart from 0 and 1) outcomes do not reappear in different states. However, we cannot apply the proof of Theorem 2 under this modification of Theorem 1 because the proof of Theorem 2 uses the structural assumption that each prize is available in each state. (This assumption is also found in the theory of Savage 1954.)

Example 3, discussed in §3, addresses problems of “pooling” opinions. There are well-studied proposals for combining the probabilities taken from several “experts” to form a single distribution that stands for the collective whole. (See Genest and Zidek 1986 for an excellent review.) However, the issue of pooling is a special case of the decision theoretic problem (discussed above), since conditional probability is a special case of state-dependent utility. That is, for a restricted set of acts, e.g., for called-off bets, the agent’s utility U_j (given state x_j) is his/her conditional probability given x_j . Thus, we investigate the adequacy of pooling rules from the standpoint of the decisions they induce. Specifically, which pooling rules satisfy the Pareto condition?

2. Existence of Compromises

Let the set of possible states of nature be \mathcal{X} , a finite set, and let the set of possible prizes be \mathcal{G} . We will not assume that the same prizes are necessarily available in every state. Hence, for each state $x \in \mathcal{X}$, we let \mathcal{G}_x be the set of prizes that are available in state x . We will let X stand for the random (unknown) state which eventually occurs. Let \mathcal{S} stand for the set of prize-state pairs $s = (g, x)$. Consider two Bayesian agents. Let agent i (for $i = 1, 2$) have subjective probability P_i (with expectations denoted E_i) and

state-dependent utility U_i , where $U_i(g, x)$ denotes the utility to agent i of reward g if state x occurs. We will denote by H a general act such that for each $x \in \mathcal{X}$, $H(x)$ is a simple von Neumann-Morgenstern lottery over \mathcal{G}_x . We suppose that agent i ranks each act H according to the value of $E_i[U_i(H(X), X)]$. We will also suppose that agent i has ranked all prize-state lotteries using U_i . That is, if Q is a stipulated probability over the set \mathcal{S} which specifies prize-state lottery \hat{L} , agent i ranks \hat{L} according to the value of $U_i(\hat{L}) = \sum_{\mathcal{S}} U_i(s)Q(s)$. Of course, we assume that the rankings of each agent satisfy Axiom 5. All probabilities in this paper are countably additive. The proofs of the new results given in this section appear in an appendix.

We now ask what are the possible representations for a Bayesian compromise between two such agents if we require that the compromise satisfy a weak Pareto condition both for acts and for prize-state lotteries.

DEFINITION 2. By a *ranking of acts*, we mean a preference relation which satisfies Axiom 1 (weak order). If l is a function from acts to real numbers, we will say that l *ranks acts* as follows: l ranks act T as weakly (strictly) preferred to act V if and only if $l(T) \geq (>)l(V)$. We say that a ranking of acts satisfies the *weak Pareto condition* with respect to two agents if, whenever $E_i[U_i(G(X), X)] < E_i[U_i(R(X), X)]$ for two acts G and R and for both $i = 1$ and $i = 2$, R ranks higher than G . Similarly, a ranking of prize-state lotteries satisfies the *weak Pareto condition* with respect to two agents if, whenever $U_i(\hat{L}_1) < U_i(\hat{L}_2)$ for two prize-state lotteries \hat{L}_1 and \hat{L}_2 and for both $i = 1$ and $i = 2$, \hat{L}_2 ranks higher than \hat{L}_1 . We say that a ranking of acts and a ranking of prize-state lotteries are *Bayesian* if they each satisfy Axioms 1–3 and together they satisfy Axiom 5.

It is clear that every probability/utility pair (P, U) provides a ranking of acts by expected utility as follows. Let E mean expectation. Then set $l(T) = E(U(T(X), X))$. This leads to the following definition.

DEFINITION 3. Let P' and P'' be probabilities with corresponding expectations denoted E' and E'' . Let U' and U'' be utilities. We say that $E'U'$ *ranks all acts the same as* $E''U''$ if, for every pair of acts T and V , $E'[U'(T(X), X)] \leq E'[U'(V(X), X)]$ if and only if $E''[U''(T(X), X)] \leq E''[U''(V(X), X)]$.

Similarly, every utility U provides a ranking of prize-state lotteries by the values of $U(\hat{L})$.

Our results will concern two Bayesian agents who rank acts according to expected utility and rank prize-state lotteries by utility. That is, we assume that two agents have probabilities P_1 and P_2 , respectively, and utilities U_1 and U_2 , respectively. We will let expectations be denoted E_1 and E_2 , respectively. First, we recall a result of Harsanyi (1955) (see also Fishburn 1984), which says that a ranking of acts satisfies the weak Pareto condition with respect to two agents if and only if it ranks all acts the same as a convex combination of the two expected utilities of the agents.

THEOREM 3 (Harsanyi 1955). *A ranking of acts satisfies the weak Pareto condition with respect to our two agents if and only if it ranks acts the same way as*

$$l_\alpha(T) = \alpha E_1(U_1(T(X), X)) + (1 - \alpha) E_2(U_2(T(X), X)), \tag{2}$$

for some $0 \leq \alpha \leq 1$.

A corollary to this theorem contains an important ingredient of our results.

COROLLARY 1. *For $i = 1, 2$ and each $x \in X$, let $f_i(x) = P_i(X = x)$. Let the probability P (with corresponding expectation E) be defined by $P = 0.5(P_1 + P_2)$. Then $f(x) = P(X = x) = 0.5[f_1(x) + f_2(x)]$. For each α , the utility*

$$U_\alpha(g, x) = \alpha \frac{f_1(x)}{f(x)} U_1(g, x) + (1 - \alpha) \frac{f_2(x)}{f(x)} U_2(g, x), \tag{3}$$

has the property that EU_α ranks all acts the same as does l_α from (2).

In words, a ranking of acts satisfies the weak Pareto condition with respect to our two agents if and only if it ranks all acts the same as EU_α for some α .

Next, we derive a similar result for prize-state lotteries. The following lemma is essentially the same as Theorem 3, except that it refers to prize-state lotteries. (Its proof will not be given because it is identical to that of Theorem 3.)

LEMMA 1. *A utility U ranks prize-state lotteries in such a way that it satisfies the weak Pareto condition with respect to our two agents if and only if there exist $a > 0$, b , and $0 \leq \beta \leq 1$ such that $aU + b = \beta U_1 + (1 - \beta)U_2$.*

We will refer to the convex combination $\beta U_1 + (1 - \beta)U_2$ as $U^{(\beta)}$. Theorem 2 of Schervish, Seidenfeld, and Kadane (1990) says that, in order for the ranking of prize-state lotteries given by $U^{(\beta)}$ to be consistent (Axiom 5) with a ranking of acts by expected utility, there must exist a probability P_* (with expectation E_*) such that the ranking of acts is given by $E_*U^{(\beta)}$.

We are now prepared to put these results together. The question of what nonautocratic Bayesian Pareto compromises exist for both acts and prize-state lotteries becomes the question of what probabilities P_* (with expectation E_*) and which β and α exist such that $E_*U^{(\beta)}$ ranks acts the same as EU_α . We propose to answer this question by fixing $0 < \beta < 1$ and determining under what conditions there exist α and P_* such that $E_*U^{(\beta)}$ ranks acts the same as EU_α .

The next theorem contains the answer to our main question. We first offer some intuition as to the meaning of the conditions of the theorem. (The notation in this discussion includes notation introduced in the theorem.) The set B_1 can be thought of as the set of states such that the conditional probabilities given B_1 are equal. That is, $P_1(A|B_1) = P_2(A|B_1)$. We know when two agents agree on the probabilities of events, there are nonautocratic Pareto compromises available. The set B_2 is the set of states x such that the two utility functions $U_1(\cdot, x)$ and $U_2(\cdot, x)$ are essentially the same (when $c(x)$ and $d(x)$ have the same sign). We know that when agents agree on the utilities of all lotteries, nonautocratic Pareto compromises are available. By assuming that $P(B_1 \cup B_2) = 1$, we assume that almost surely one of the two cases just described occurs (i.e., for each state, the agents either agree on the probabilities or they agree on the utilities). When $c(x)$ and $d(x)$ have opposite signs, then the two utilities are in complete opposition. It is known that nonautocratic Pareto compromises exist in this case also. (See footnote 11 in Seidenfeld, Kadane, and Schervish 1989a). Note that the function r in Theorem 4 is defined in terms of the functions c and d , which may not be unique. For example, if there exists a set of x values in B_2 such that both $U_1(g, x)$ and $U_2(g, x)$ are constant in g then both $c(x)$ and $d(x)$ are arbitrary for such x values. This means that there might be many functions r of the form specified. Each such function will be called a *version* of r . The reader should also note that, if $x \in B_1 \cap B_2$, then both forms of $r(x)$ in Theorem 4 are the same.

THEOREM 4. *Let $0 < \beta < 1$. Define*

$$B_1 = \{x : \alpha(1 - \beta)f_1(x) = \beta(1 - \alpha)f_2(x)\},$$

$$B_2 = \{x : \exists b(x), c(x), d(x) \text{ such that}$$

$$d(x)U_2(g, x) = c(x)U_1(g, x) + b(x) \forall g \in \mathcal{G}_x, \text{ with not both } c(x) = 0 \text{ and } d(x) = 0\},$$

$$r(x) = \begin{cases} \frac{\alpha}{\beta} f_1(x) & \text{if } x \in B_1, \\ \frac{d(x)\alpha f_1(x) + c(x)(1 - \alpha)f_2(x)}{d(x)\beta + c(x)(1 - \beta)} & \text{if } x \in B_2 \setminus B_1. \end{cases}$$

Then $E_*U^{(\beta)}$ ranks all acts the same as EU_α if and only if the following conditions hold:

1. $P(B_1 \cup B_2) = 1$.
2. There exists a version of r such that $P(\{x : r(x) \geq 0\}) = 1$, and $f_*(x) = r(x)/a$ is the mass function of P_* , where $a = \sum_{y \in \mathcal{X}} r(y)$.

Speaking very loosely, Theorem 4 says that there exists a Bayesian Pareto compromise between two agents which applies to both acts and prize-state lotteries if and only if, for each state, either the agents agree on the probability of the state or they agree on the utility in that state. The proof of this theorem appears in the appendix. We illustrate the theorem with several examples in §3.

3. Examples

Example 1 is the special case handled by Seidenfeld, Kadane, and Schervish (1989a).

EXAMPLE 1. Suppose that $\mathcal{X} = \{0, 1\}$ and $\mathcal{G}_x = \{0, r, 1\}$ for all x . Suppose that for each i , $P_i(0) = p_i$, $U_i(0, x) = 0$ for all x , and $U_i(1, x) = 1$ for all x . Let $U_1(r, x) = r_1$ and $U_2(r, x) = r_2$. In this way, utilities are state-independent. Suppose that $p_1 \neq p_2$ and $r_1 \neq r_2$. The set B_2 is clearly empty. The set B_1 can contain at most one of the two x values, and then only for exceptional values of α and β . The result is that $P(B_1 \cup B_2) < 1$. We already know that only autocratic Pareto compromises are available in this case, and Theorem 4 confirms this.

Example 2 shows how all of the conditions of Theorem 4 can be met with both B_1 and B_2 nonempty.

EXAMPLE 2. Let $\mathcal{X} = \{1, 2, 3, 4, 5\}$ and $\mathcal{G}_x = \{g_0, g_1, g_2\}$ for all x . Let the state-dependent utilities be

$$\begin{aligned}
 U_1(g_j, 1) &= j, & U_2(g_j, 1) &= 2j, \\
 U_1(g_j, 2) &= j, & U_2(g_j, 2) &= \frac{1}{2}j, \\
 U_1(g_j, 3) &= 1, & U_2(g_j, 3) &= j, \\
 U_1(g_j, 4) &= j, & U_2(g_j, 4) &= 2 - j, \\
 U_1(g_j, 5) &= j, & U_2(g_j, 5) &= \begin{cases} 0 & \text{if } j = 0, \\ 2 & \text{if } j = 1, \\ 1 & \text{if } j = 2. \end{cases}
 \end{aligned}$$

The two probabilities are $P_2(x) = 0.2$ for all x and $P_1(x) = 0.31$ for $x \in \{1, 2, 3\}$, $P_1(4) = 0.02$, $P_1(5) = 0.05$. With $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$, we see that $B_1 = \{5\}$ and $B_2 = \{1, 2, 3, 4\}$. The values of $c(x)$ and $d(x)$ are

$$c(x) = \begin{cases} 2 & \text{if } x = 1, \\ 1 & \text{if } x = 2, \\ \text{arbitrary} & \text{if } x = 3, \\ 1 & \text{if } x = 4, \end{cases}$$

$$d(x) = \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x = 2, \\ 0 & \text{if } x = 3, \\ -1 & \text{if } x = 4. \end{cases}$$

We calculate $r(x)$ as

$$r(x) = \begin{cases} 0.204 & \text{if } x = 1, \\ 0.36 & \text{if } x = 2, \\ 0.1 & \text{if } x = 3, \\ 0.16 & \text{if } x = 4, \\ 0.1 & \text{if } x = 5. \end{cases}$$

The sum of these values is $a = 0.924$. It follows that the mass function of P_* is

$$f_*(x) = \begin{cases} 0.2208 & \text{if } x = 1, \\ 0.3896 & \text{if } x = 2, \\ 0.1082 & \text{if } x = 3, \\ 0.1732 & \text{if } x = 4, \\ 0.1082 & \text{if } x = 5. \end{cases}$$

To verify that everything worked out, we tabulate both $U^{(\beta)}(g, x)$ and $U_\alpha(g, x)$ in the format $(U(g_1, x), U(g_2, x), U(g_3, x))$:

x	$af_*(x)U^{(\beta)}(\cdot, x)$	$U_\alpha(\cdot, x)$
1	(0, 0.34, 0.68)	(0, 0.34, 0.68)
2	(0, 0.24, 0.48)	(0, 0.24, 0.48)
3	(0.0333, 0.1, 0.1666)	(0.2066, 0.2733, 0.34)
4	(0.2133, 0.16, 0.1066)	(0.1333, 0.08, 0.266)
5	(0, 0.1666, 0.1333)	(0, 0.1666, 0.1333)

We see that the difference between $af_*(x)U^{(\beta)}(g, x)$ and $U_\alpha(g, x)$ is constant in g (although not constant in x). Hence, $E_*U^{(\beta)}$ ranks all acts the same as EU_α .

Example 3 shows how the results of this paper are relevant to considerations of “externally Bayesian” pooling operators.

EXAMPLE 3. Suppose that interest lies in the joint distribution of two random quantities (X, G) lying in a finite space $\mathcal{X} \times \mathcal{G}$ as well as in the conditional distribution of G given X . Suppose we have two probability functions P_1 and P_2 over $\mathcal{X} \times \mathcal{G}$. For each $0 \leq \beta \leq 1$, let $P_\beta = \beta P_1 + (1 - \beta)P_2$, which is called the *linear opinion pool*. Let $P_i^{G|X}$ denote the conditional mass function of G given X derived from P_i , and let P_i^X denote the marginal mass function of X . Similarly, let $P_\beta^{G|X}$ and P_β^X denote the conditional and marginal mass functions derived from P_β . The linear opinion pool is *externally Bayesian* if

$$P_\beta^{G|X} = \beta P_1^{G|X} + (1 - \beta)P_2^{G|X}. \tag{4}$$

That is to say, if we pool the joint distributions and then condition on X , we get the same result as if we condition each distribution on X and then pool.

An interesting question arises as to whether a linear opinion pool can be externally Bayesian. Theorem 4 provides an answer. Let the set of prizes be \mathcal{G} and let the set of states be \mathcal{X} . Define $U_i(g, x) = P_i^{G|X}(g|x)$, that is, let the conditional mass function of G given $X = x$ play the role of the state-dependent utility function. Also, let the marginal distribution of X play the role of the probability. The right-hand side of (4) is just $U^{(\beta)}$. The left-hand side of (4) can be written as

$$\begin{aligned}
 P_\beta^{G|X} &= \frac{\beta P_1^{G|X} P_1^X + (1 - \beta) P_2^{G|X} P_2^X}{\beta P_1^X + (1 - \beta) P_2^X} \\
 &= \frac{U_\beta}{P_*}, \quad \text{where} \\
 P_* &= \beta P_1^X + (1 - \beta) P_2^X. \tag{5}
 \end{aligned}$$

It follows that (4) holds if and only if $U_\beta = P_* U^{(\beta)}$. Using this correspondence, an act H would correspond to a stipulated probability distribution over \mathcal{G} for each $x \in \mathcal{X}$, say $P_H(\cdot | x)$. Then

$$E_i U_i(H, X) = \sum_x \sum_g U_i(g, x) P_i^X(x) P_H(g|x),$$

and, if we let E_* be the expectation corresponding to P_* ,

$$E_* U^{(\beta)}(H, X) = \sum_x \sum_g P_*(x) U^{(\beta)}(g, x) P_H(g|x).$$

It follows that (4) holds if and only if $E_* U^{(\beta)}$ ranks all acts the same as $\beta E_1 U_1 + (1 - \beta) E_2 U_2$. This, in turn, holds if and only if the conditions of Theorem 4 hold. Condition 1 says that both P_1^X and P_2^X must assign probability 1 to the union of the two sets B_1 and B_2 . Since $\alpha = \beta$ in this case, we can write

$$B_1 = \{x : P_1^X(x) = P_2^X(x)\}.$$

Condition 2 says that P_* must equal P_i^X/a on B_1 . But (5) implies that $a = 1$. On $B_2 \setminus B_1$, $P_*(x)$ must be a convex combination of $P_1^X(x)$ and $P_2^X(x)$, but it will be a different convex combination than (5) unless $b(x) = 0$ and $c(x) = d(x) = 1$. It also follows that

$$B_2 = \{x : P_1^{G|X}(g|x) = P_2^{G|X}(g|x), \text{ for all } g\}.$$

In summary, (4) holds if and only if, for almost all x , either $P_1^X(x) = P_2^X(x)$ or $P_1^{G|X}(\cdot | x) = P_2^{G|X}(\cdot | x)$.

4. Conclusion

Theorem 4 shows that, for two Bayesian agents, nonautocratic (weak) Pareto compromises exist only under very restrictive conditions. This result extends Theorem 2 by allowing for state-dependent utilities. Our results suggest that, even with state-dependent utilities, there is little hope that two Bayesians can arrive at a Bayesian compromise which satisfies the (weak) Pareto rule.

We see three ways for avoiding the unpleasant conclusion that Bayesians cannot find cooperative Bayesian compromises:

1. Savage (1954, §13.5), in his discussion of the group minimax-regret rule, concludes that the standards of rational group behavior need not be the same as the standards of his theory of rational individual behavior. Specifically, he offers that the rational individual, but not the group, ought to be committed to the principle that preference is a weak-order (his postulate P1).

2. Levi (1982) argues for a unified account of individual and (cooperative) group decision making—without the assumption that preference induces a weak-order. But, in rebuttal to our Theorem 2, he finds that the (weak) Pareto condition is unwarranted (see Levi 1990). Roughly put, he defends a logically weaker rule, which he calls “Robust Pareto,” wherein the (weak) Pareto condition applies only when, e.g., each agent’s preference, for, say H_2 over H_1 , is invariant over an interchange of their probabilities (for states), or is invariant over an interchange of their utilities (for outcomes). His position is that the preference relation, as captured by expected utility inequalities, is a derivative

notion to be supported by reasons, expressed in terms of probabilities and utilities. Unless the agents can find common reasons for their common preferences, such preferences are *not* to count in a Pareto compromise. The Robust Pareto rule is intended to capture just those cases where common preferences are supported by some common reasons. Under this modification of Pareto, using Levi's Robust Pareto rule, there exists a convex family of Bayesian compromises for each pair of Bayesian agents. Hence, in Levi's theory, there is no result analogous to our Theorems 2 and 4.

3. In Seidenfeld, Kadane, and Schervish (1989b) and Seidenfeld, Schervish, and Kadane (1990), we explore representations for a theory of preference, where preference is a strict partial order, using sets of probability/utility pairs. This provides a unified standard of rational behavior across individuals and cooperative groups, and yet maintains the (weak) Pareto principle. We hope that approach will afford a viable solution to the challenge of rational group behavior.¹

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Appendix. Proof of Theorem 4

The proof of Theorem 4 will proceed through several lemmas. We will use the following notation throughout this appendix. Probabilities on \mathcal{X} will be denoted by P with superscripts or subscripts and the corresponding mass functions and expectations will be denoted by the letters f and E , respectively, with the same superscripts or subscripts. For example, $f''(x) = P''(X = x)$ and $E''(h(X)) = \sum_{x \in \mathcal{X}} h(x)f''(x)$.

LEMMA 2. *Suppose $E'U'$ and $E''U''$ rank all acts the same. Then there exist $a > 0$ and b , such that, for every act G ,*

$$E'[aU'(G(X), X) + b] = E''[U''(G(X), X)]. \tag{6}$$

PROOF. First, suppose that there exist two acts T and R such that $E'[U'(R(X), X)] < E'[U'(T(X), X)]$ and $E''[U''(R(X), X)] < E''[U''(T(X), X)]$. Then set

$$a = \frac{E''[U''(T(X), X)] - E''[U''(R(X), X)]}{E'[U'(T(X), X)] - E'[U'(R(X), X)]},$$

$$b = E''[U''(R(X), X)] - aE'[U'(R(X), X)].$$

It follows that both of the next equations hold

$$E'[aU'(R(X), X) + b] = E''[U''(R(X), X)],$$

$$E'[aU'(T(X), X) + b] = E''[U''(T(X), X)].$$

For each act G , there exists $0 \leq \alpha \leq 1$ such that one of the following is true:

$$\alpha T + (1 - \alpha)R \approx G,$$

$$\alpha G + (1 - \alpha)R \approx T,$$

$$\alpha T + (1 - \alpha)G \approx R,$$

where \approx refers to the common preference ranking of $E'U'$ and $E''U''$. In the first case, both

$$E'[U'(G(X), X)] = \alpha E'[U'(T(X), X)] + (1 - \alpha)E'[U'(R(X), X)],$$

$$E''[U''(G(X), X)] = \alpha E''[U''(T(X), X)] + (1 - \alpha)E''[U''(R(X), X)],$$

and we see that (6) holds. The proof is similar in the other two cases.

Finally, suppose that both $E'U'$ and $E''U''$ rank all acts as equivalent. Then, for all acts G , $E'[U'(G(X), X)] = c'$, say and $E''[U''(G(X), X)] = c''$, say. Let $a = 1$ and $b = c'' - c'$ to complete the proof. \square

Under the conditions of Lemma 2, the following lemma allows us to replace the two different probabilities with a single probability. (The proof is trivial and not given.)

LEMMA 3. Suppose that $E[U'(G(X), X)] = E''[U''(G(X), X)]$ for every act G . Let P be a probability (with mass function f) such that $f(x) = 0$ implies $f'(x) = 0$ and $f''(x) = 0$. Then

$$E\left[\frac{f'(X)}{f(X)} U'(G(X), X)\right] = E\left[\frac{f''(X)}{f(X)} U''(G(X), X)\right]$$

for every act G .

The following lemma says that, under the conditions of Lemmas 2 and 3, for each x , the two utilities U' and U'' , as functions of g , must be related by an affine transformation.

LEMMA 4. If $E[U'(G(X), X)] = E[U''(G(X), X)]$ for every act G , then $P\{x : U'(g, x) - U''(g, x) \text{ is constant in } g\} = 1$.

PROOF. Let

$$C_+ = \{x : \exists g(x), h(x) \in \mathcal{G}_x \text{ such that } U'(g(x), x) - U''(g(x), x) > U'(h(x), x) - U''(h(x), x)\},$$

$$C_- = \{x : \exists g(x), h(x) \in \mathcal{G}_x \text{ such that } U'(g(x), x) - U''(g(x), x) < U'(h(x), x) - U''(h(x), x)\}.$$

We need to prove that $P(C_+ \cup C_-) = 0$. Define two acts

$$G(x) = \begin{cases} g(x) & \text{if } x \in C_+, \\ \text{arbitrary} & \text{if } x \notin C_+, \end{cases}$$

$$H(x) = \begin{cases} h(x) & \text{if } x \in C_+, \\ G(x) & \text{if } x \notin C_+. \end{cases}$$

For $x \in C_+$, set

$$r(x) = U'(g(x), x) - U''(g(x), x) - U'(h(x), x) + U''(h(x), x),$$

which is strictly positive for all $x \in C_+$. Set $r(x) = 0$ for $x \notin C_+$. It is easy to see that

$$0 = E[U'(G(X), X)] - E[U''(G(X), X)] - E[U'(H(X), X)] + E[U''(H(X), X)] = E[r(X)],$$

which implies that $P(C_+) = 0$. A similar proof shows $P(C_-) = 0$. \square

We are now in position to prove Corollary 1.

PROOF OF COROLLARY 1. For every act G ,

$$\alpha E_1[U_1(G(X), X)] + (1 - \alpha) E_2[U_2(G(X), X)]$$

$$= E[\alpha f_1(X) U_1(G(X), X) + (1 - \alpha) f_2(X) U_2(G(X), X)]. \quad \square$$

In the remainder of the theorems, U_α will always mean (3), and P will mean $0.5(P_1 + P_2)$.

LEMMA 5. Suppose that $E'U'$ ranks all acts the same as EU_α . The following are true:

- There exists P'' such that $E''U'$ ranks acts the same as $E'U'$ and $f(x) = 0$ implies $f''(x) = 0$.
- There exists a $a > 0$ such that

$$P\left(x : a \frac{f''(x)}{f(x)} U'(g, x) - U_\alpha(g, x) \text{ is constant in } g\right) = 1.$$

For each x , call the constant value $v_\alpha(x)$.

PROOF. If $f(x) = 0$ implies $f''(x) = 0$, we need only prove the second part of the lemma. Let A be the set of all x^* such that $f(x^*) = 0$ but $f''(x^*) > 0$. It must be that

$$P'(x \in A : U'(g, x) \text{ is constant in } g) = P'(A),$$

or else $E'U'$ would be able to distinguish acts that differed only on A , while EU_α would not. Suppose that $c(x)$ is the function such that $P'(\{x : U'(g, x) = c(x) \text{ for all } g\}) = P'(A)$. There are two cases to consider.

1. First, if $P'(A) = 1$, then $E'U'$ must rank all acts as equivalent because $E'(U'(G(X), X)) = E'(c(X))$ for every act G . In this case, set $P'' = P$. Then $E''U'$ will also rank all acts as equivalent.

2. Second, suppose $0 < P'(A) < 1$. For each event B define $P''(B) = P'(B|A^c)$. Now $f(x) = 0$ implies $f''(x) = 0$, and for each act G

$$E''U'(G, X) = P'(A^c)E''U'(G, X) + E'(c(X)).$$

It is now clear that $E''U'$ ranks all acts the same as $E'U'$.

For the second part of the lemma, use Lemma 2 to find $a > 0$ and b such that, for every act G ,

$$E[U_\alpha(G(X), X)] = E^u[aU'(G(X), X) + b] = E\left[a \frac{f''(X)}{f(X)} U(G(X), X) + b\right].$$

Next, use Lemma 4 to conclude that

$$P\left(x : a \frac{f''(x)}{f(x)} U(g, x) + b - U_\alpha(g, x) \text{ is constant in } g\right) = 1. \quad \square$$

In light of Lemma 5, we will assume that each P_* that we consider satisfies “ $f(x) = 0$ implies $f_*(x) = 0$.”

LEMMA 6. $E_*U^{(\beta)}$ ranks all acts the same as EU_α if and only if there exists $a > 0$ such that

$$h_a(g, x) = [a\beta f_*(x) - \alpha f_1(x)]U_1(g, x) + [a(1 - \beta)f_*(x) - (1 - \alpha)f_2(x)]U_2(g, x) \quad (7)$$

is constant in g a.s. $[P]$. Call this constant value $h_a(x)$.

PROOF. For the “if” part, we note that, for every act G ,

$$aE_*[U^{(\beta)}(G(X), X)] - E[U_\alpha(G(X), X)] = E\left[\frac{h_a(X)}{f(X)}\right],$$

which is the same for all acts because we assume (7). It follows that $E_*U^{(\beta)}$ ranks acts the same as EU_α . For the “only if” part, apply Lemma 5. \square

We are now in position to prove the “if” part of Theorem 4.

PROOF (of the “if” part of Theorem 4). Assume that conditions 1 and 2 of Theorem 4 hold. For all $x \in B$,

$$a\beta f_*(x) = \alpha f_1(x),$$

$$a(1 - \beta)f_*(x) = (1 - \alpha)f_2(x).$$

It follows that $h_a(g, x)$ from (7) equals 0 for all $x \in B_1$. For $x \in B_2$ and such that $d(x) \neq 0$,

$$\begin{aligned} h_a(g, x) &= [\beta r(x) - \alpha f_1(x)]U_1(g, x) + [(1 - \beta)r(x) - (1 - \alpha)f_2(x)]\left[\frac{c(x)U_1(g, x) + b(x)}{d(x)}\right] \\ &= \frac{b(x)}{d(x)} [(1 - \beta)r(x) - (1 - \alpha)f_2(x)] \\ &= \frac{b(x)}{\beta d(x) + (1 - \beta)c(x)} [\alpha(1 - \beta)f_1(x) - (1 - \alpha)\beta f_2(x)], \end{aligned}$$

which is constant in g . Similarly, if $c(x) \neq 0$,

$$\begin{aligned} h_a(g, x) &= \left[\beta \frac{d(x)\alpha f_1(x) + c(x)(1 - \alpha)f_2(x)}{d(x)\beta + c(x)(1 - \beta)} - \alpha f_1(x)\right]\left[\frac{d(x)U_2(g, x) - b(x)}{c(x)}\right] \\ &\quad + \left[(1 - \beta) \frac{d(x)\alpha f_1(x) + c(x)(1 - \alpha)f_2(x)}{d(x)\beta + c(x)(1 - \beta)} - (1 - \alpha)f_2(x)\right]U_2(g, x) \\ &= -\frac{b(x)}{c(x)} [\beta r(x) - \alpha f_1(x)] \\ &= \frac{b(x)}{\beta d(x) + (1 - \beta)c(x)} [\alpha(1 - \beta)f_1(x) - (1 - \alpha)\beta f_2(x)], \end{aligned}$$

which is the same as when $d(x) \neq 0$. So, the conditions of Lemma 6 (the “if” part) are met and we conclude that $E_*U^{(\beta)}$ ranks all acts the same as EU_α . \square

LEMMA 7. Let B_1 and B_2 be as in Theorem 4, and suppose that $P(B_1 \cup B_2) = 1$. Suppose also that there exist P_* and α such that $E_*U^{(\beta)}$ ranks all acts the same as EU_α . Then $f_* = dP_*/dP$ must have the form given in Theorem 4, namely $r(x)/a$, for some version of r .

PROOF. First, we look at $x \in B_2$. Lemma 6 says that $h_a(g, x)$ must be constant in g . We deal with four cases here.

Case (i). Both $U_1(g, x)$ and $U_2(g, x)$ are constant in g . Then $h_a(g, x)$ is constant in g , and both $c(x)$ and $d(x)$ can be arbitrary (so long as both are not 0). This means that $r(x)$ is arbitrary, and every $f_*(x)$ has the desired form.

Case (ii). $U_1(g, x)$ is constant in g , but $U_2(g, x)$ is not. In this case $d(x) = 0$ is necessary, and $r(x) = (1 - \alpha)f_2(x)/(1 - \beta)$. It is also true in this case that $h_a(g, x)$ equals a constant plus $[af_*(x)(1 - \beta) - (1 - \alpha)f_2(x)]U_2(g, x)$. For this to be constant, it is necessary that $f_*(x) = r(x)/a$.

Case (iii). $U_2(g, x)$ is constant in g , but $U_1(g, x)$ is not. This is virtually the same as the previous case.

Case (iv). Neither $U_1(g, x)$ nor $U_2(g, x)$ is constant in g . In this case, neither $c(x) = 0$ nor $d(x) = 0$ is possible. It also follows that $c(x)$ and $d(x)$ are unique up to multiplication by nonzero constant (for each x). To see this, suppose that for all g , $c_j(x)U_1(g, x) + b_j(x) = d_j(x)U_2(g, x)$ for $j = 1, 2$. Set $c^*(x) = c_2(x)/c_1(x)$. Then

$$c^*(x)b_1(x) - b_2(x) = [c^*(x)d_1(x) - d_2(x)]U_2(g, x).$$

It is clear that $c^*(x)d_1(x) = d_2(x)$, and we have proven uniqueness. It follows that the form of $r(x)$ is unique in this case. Now, write $h_a(g, x)$ as

$$\begin{aligned} & [af_*(x)\beta - \alpha f_1(x)]U_1(g, x) + [af_*(x) - (1 - \alpha)f_2(x)] \frac{c(x)U_1(g, x) - b(x)}{d(x)} \\ &= U_1(g, x) \left[af_*(x)\beta - \alpha f_1(x) + \frac{c(x)}{d(x)}af_*(x)(1 - \beta) - \frac{c(x)}{d(x)}(1 - \alpha)f_2(x) \right] \\ & \qquad \qquad \qquad - \frac{b(x)}{d(x)} [af_*(x) - (1 - \alpha)f_2(x)]. \end{aligned}$$

This is constant in g if and only if $f_*(x) = r(x)/a$.

Next, look at $x \in B_1 \setminus B_2$. The only way for $h_a(g, x)$ to be constant in g is for the coefficients of both $U_1(g, x)$ and $U_2(g, x)$ to be zero. The reason is that, if not, x would be in B_2 . So, $af_*(x)\beta = \alpha f_1(x)$ and we see that $f_*(x) = r(x)/a$. \square

LEMMA 8. Let B_1 and B_2 be as in Theorem 4. Suppose that P_* and α exist such that $E_*U^{(\beta)}$ ranks all acts the same as EU_α . Then $P(B_1 \cup B_2) = 1$.

PROOF. Suppose that such a P_* and α exist. Let $a > 0$ and b be as guaranteed by Lemma 2. Let $B_3 = (B_1 \cup B_2)^c$. Then, for every $x \in B_3$ and for every $d(x)$ and $c(x)$, there exist $g_1(x), g_2(x) \in \mathcal{G}_x$ such that

$$d(x)U_2(g_1(x), x) - c(x)U_1(g_1(x), x) > d(x)U_2(g_2(x), x) - c(x)U_1(g_2(x), x), \tag{8}$$

and $\alpha(1 - \beta)f_1(x) \neq \beta(1 - \alpha)f_2(x)$. So, for each $x \in B_3$, define

$$\begin{aligned} c(x) &= af_*(x)\beta - \alpha f_1(x), \\ d(x) &= af_*(x)(1 - \beta) - (1 - \alpha)f_2(x). \end{aligned}$$

Both $c(x) = 0$ and $d(x) = 0$ simultaneously is not possible, since this would imply that $x \in B_1$. For these choices of $c(x)$ and $d(x)$, define $g_1(x)$ and $g_2(x)$ to satisfy (8). Now, define two acts

$$\begin{aligned} G_1(x) &= \begin{cases} g_1(x) & \text{if } x \in B_3, \\ \text{arbitrary} & \text{if } x \notin B_3, \end{cases} \\ G_2(x) &= \begin{cases} g_2(x) & \text{if } x \in B_3, \\ G_1(x) & \text{if } x \notin B_3. \end{cases} \end{aligned}$$

We now have that

$$\begin{aligned} 0 &= E_*[aU^{(\beta)}(G_1(X), X)] - E[U_\alpha(G_1(X), X)] - \{E_*[aU^{(\beta)}(G_2(X), X)] - E[U_\alpha(G_2(X), X)]\} \\ &= \sum_{x \in B_3} [af_*(x)\beta - \alpha f_1(x)][U_1(g_1(x), x) - U_1(g_2(x), x)] \\ & \quad + [af_*(x)(1 - \beta) - (1 - \alpha)f_2(x)][U_2(g_1(x), x) - U_2(g_2(x), x)] \\ &= \sum_{x \in B_3} [c(x)U_1(g_1(x), x) - d(x)U_2(g_1(x), x)] - [c(x)U_1(g_2(x), x) - d(x)U_2(g_2(x), x)]. \end{aligned}$$

By (8), each term in this last sum is positive, hence B_3 must be empty. \square

Finally, we are ready to prove the "only if" part of Theorem 4.

PROOF (of the “only if” part of Theorem 4). Suppose that there exists a P_* as in the statement of the theorem. We will now prove that conditions 1 and 2 hold. Lemma 8 says that if such a P_* exists, then condition 1 holds. Lemma 7 says that if such a P_* exists, then condition 2 holds. \square

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